

# HOW TO INTEGRATE DIVERGENT INTEGRALS: A PURE NUMERICAL APPROACH TO COMPLEX LOOP CALCULATIONS

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## **Abstract**

Loop calculations involve the evaluation of divergent integrals. Usually[1] one computes them in a number of dimensions different than four where the integral is convergent and then one performs the analytical continuation and considers the Laurent expansion in powers of  $\varepsilon = n - 4$ . In this paper we discuss a method to extract directly all coefficients of this expansion by means of concrete and well defined integrals in a five dimensional space. We bypass the formal and symbolic procedure of analytic continuation; instead we can numerically compute the integrals to extract directly both the coefficient of the pole  $1/\varepsilon$  and the finite part.

# Introduction

Feynman diagrams computations are often a lengthy and hard task. The complexity of typical computations grows as a factorial with the number of external legs and/or loops involved in the processes. In the Standard Model the main difficulties arise from QCD corrections and the production of jets in the final states; but it is not difficult to imagine extensions of the Standard Model, in which this problem could seriously limit our capabilities of studying and understanding new physical phenomena; in particular if they are involved (in addition to  $\alpha_s$ ) by new rather strong couplings, which give rise to multiparticle/multiloop amplitudes. This issue motivates a rich and increasing activity. Considerable progresses have been achieved in the past [2]: new methods inspired by string theory [3-5], helicity amplitudes [6-9], different recursive algorithms to calculate QCD dual amplitudes[10], or the scattering matrix elements of generic processes with arbitrary initial/final states[11]. Very useful approximations have been proposed [12-16], when the exact matrix elements are unknown. In loop computations interesting simplifications [17-26] can be used.

While analytical methods can provide us in some specific processes with very simple results, a general approach for generic lagrangian and final states cannot avoid the use of the computer power. A possible strategy is to implement the Feynman rules into some automatic code with the help of some familiar packages for symbolic manipulation of algebraic formulae. However this way to proceed becomes ineffective, when the computer has to manage very complex formulae (as often it is the case).

This suggests a second strategy which tries exploiting the computer power with pure numerical algorithms and thus avoiding lengthy symbolic manipulation. In the recent past these algorithms have been successfully used for the computation of several tree level amplitudes both in QCD[27, 28] and in electroweak processes [29-31]. In this paper we discuss the possibility to apply pure numerical techniques[32] in loop calculations.

The problem of performing loop computations can be divided in two parts. The first part is similar to tree level computations[11]: from a given lagrangian one has to produce an explicit function of virtual and real momenta which is equivalent to the sum of all the Feynman diagrams contributing to the process. Once this task is accomplished, loops calculations have an additional problem with respect to tree level ones: integrals over virtual loop momenta are affected by ultra-violet and/or infrared divergences. These divergences require the introduction of a regularization procedure which usually obliges us to perform all virtual loop integrations in an analytical and essentially symbolic fashion (while in tree level calculations one usually performs the integration over the real momenta by numerical montecarlo methods). This is a serious difficulty in any numerical approach. Here we address this last issue. We will define a *numerical* procedure to perform virtual loop integrations even when these include divergences.

Let us consider a function  $f[l_\mu, p_\mu]$  of the external momenta  $p_\mu$  and of the virtual momenta  $l_\mu$ . The  $n$  dimensional integration[1] is performed in a region of complex values of  $n$  where the integral is convergent; then one considers the analytical continuation to values of  $n$  where the integral is

not well defined. In general for  $n \simeq 4$  the analytical continuation can have a pole  $1/\varepsilon$  ( $\varepsilon = n - 4$ ). We can use the Laurent expansion around  $\varepsilon \simeq 0$  to define some functionals  $I_k[f]$  such that<sup>1</sup>

$$\int d^n l f[l_\mu] = I_{-1}[f]\varepsilon^{-1} + I_0[f] + I_1[f]\varepsilon^1 + I_2[f]\varepsilon^2 + \dots \quad (1)$$

Since the  $n$  dimensional integration is well defined for any function  $f$ , also the functionals  $I_k[f]$  are unambiguously defined from equation (1). For instance,  $I_0[f]$  coincides with the ordinary four dimensional integral

$$\int d^4 \bar{l} f[l_\mu] \quad (2)$$

if  $f$  is a convergent function. While the left-hand side of the (1) is essentially a formal definition which does not lead us to an obvious numerical integration procedure, each  $I_k[f]$  can be written in terms of well defined and convergent integrals. In the following we show how to build explicit integral representations for the functionals  $I_k[f]$ . In the next sections, we explain two different definitions (even if equivalent) of the  $I_k[f]$ ; in the final part some very simple numerical examples will be given in order to emphasize the practical purposes of the method.

## Method of Integration A

Let us assume that we have to perform an integral in  $n > 4$  dimensions. The virtual momentum  $l_\mu$  has  $n$  components. We call  $\bar{l}_\mu$  the components with  $\mu \leq 4$  and  $\tilde{l}_\mu$  the components with  $\mu > 4$ . Then we can rewrite the integral

$$\int d^n l f[l_\mu, p_\mu] = \int d^4 \bar{l} f[l_\mu, p_\mu] d^\varepsilon \tilde{l} = \int d^4 \bar{l} f[l_\mu, p_\mu] \tilde{l}^{\varepsilon-1} d\tilde{l} d\Omega_\varepsilon \quad (3)$$

$d\Omega_\varepsilon$  is the solid angle in the subspace ( $\mu > 4$ ) of dimension  $\varepsilon$  orthogonal to the four-vector  $\bar{l}_\mu$ . Note that  $f[l_\mu, p_\mu]$  is invariant under rotations in this subspace, since the external momenta do not have components  $\mu > 4$ . We can exploit this invariance as follows. First we rotate the vector  $\tilde{l}_\mu$  such that  $\tilde{l}_5 \neq 0$  and  $\tilde{l}_n = 0$  for any  $n > 5$ . Thus we can write  $l_\mu = (\bar{l}_1, \bar{l}_2, \bar{l}_3, \bar{l}_4, \sqrt{t})$  with  $t = (l_5)^2 + (l_6)^2 + \dots + (l_n)^2$  and omit all the components with  $\mu > 5$ . We have reduced our  $n$  dimensional space into a five dimensional space. Second, we can perform the integral in  $d\Omega_\varepsilon$  and we obtain from eq.(3)

$$\frac{\pi^{\varepsilon/2}}{\Gamma[\frac{\varepsilon}{2}]} \int_0^\infty \left( \int d^4 \bar{l} f[l_\mu, p_\mu] \right) t^{\varepsilon/2-1} dt. \quad (4)$$

Now the integral in  $dt$  can be integrated by parts

$$\frac{\pi^{\varepsilon/2}}{\Gamma[\frac{\varepsilon}{2}]} \int_0^\infty \frac{4t^{1+\frac{\varepsilon}{2}}}{\varepsilon(2+\varepsilon)} \left( \int d^4 \bar{l} \frac{d^2}{dt^2} f[l_\mu, p_\mu] \right) dt. \quad (5)$$

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<sup>1</sup>At one loop.

The function  $f$  depends on  $t$  through the fifth component of  $l_\mu$ . The function  $t^{1+\frac{\varepsilon}{2}}$  has a cut in the positive real axis of  $t$  ( $Re(t) > 0$  and  $Im(t) = 0$ ), in fact the rotation  $t \rightarrow te^{i2\pi}$  gives  $t^{1+\frac{\varepsilon}{2}} \rightarrow t^{1+\frac{\varepsilon}{2}+i\pi\varepsilon}$ . Therefore we can apply the following identity

$$\int_0^\infty t^{1+\frac{\varepsilon}{2}} f[t] dt = \frac{1}{(1 - e^{i\pi\varepsilon})} \oint t^{1+\frac{\varepsilon}{2}} f[t] dt \quad (6)$$

where the contour of the last integration in  $t$  must contain all the poles and singularities lying in the complex  $t$  plane and must avoid the cut on the positive real axis (see figure 1). Then the integral (5) above becomes

$$\frac{1}{(1 - e^{i\pi\varepsilon})} \frac{\pi^{\varepsilon/2}}{\Gamma[\frac{\varepsilon}{2}]} \oint \frac{4t^{1+\frac{\varepsilon}{2}}}{\varepsilon(2+\varepsilon)} \left( \int d^4\bar{l} \frac{d^2}{dt^2} f[l_\mu, p_\mu] \right) dt. \quad (7)$$

Now we can make the expansion in powers of  $\varepsilon$  (the Euler constant and  $\log(\pi)$  have been reabsorbed into a redefinition of the scale  $\mu$  as in the  $\overline{MS}$ ) and we obtain the final result

$$\oint \frac{it}{2\pi} \left( \frac{2}{\varepsilon} + (-1 + \log[-t]) \right) \left( \int d^4\bar{l} \frac{d^2}{dt^2} f[l_\mu, p_\mu] \right) dt. \quad (8)$$

In the above expression each integral is now convergent, due to the action of the derivative  $d^2/dt^2$  appearing inside the integral in  $d^4\bar{l}$ . Thus one can separately calculate the singular and the non singular part, simply evaluating well defined and convergent integrals. We can check the result above in a specific example and see how the above formula can be used for practical calculations.

Suppose that we have to evaluate the following divergent integral

$$\int d^n l \frac{l^2}{l^2 + m^2}. \quad (9)$$

In our approach  $l_\mu$  is not a  $n$  dimensional object but it is a more concrete five dimensional vector with components  $(\bar{l}_1, \bar{l}_2, \bar{l}_3, \bar{l}_4, \sqrt{t})$ ; thus  $l^2 = l_\mu l^\mu = \bar{l}^2 + t$  with  $\bar{l}^2$  equal to the squared length of the real four-vector  $(\bar{l}_1, \bar{l}_2, \bar{l}_3, \bar{l}_4)$  and  $t$  a complex number. If we are interested in the finite part, we can apply the non singular term in equation (8) and we have

$$\oint \frac{it}{2\pi} (-1 + \log[-t]) \left( \frac{d^2}{dt^2} \frac{\bar{l}^2 + t}{\bar{l}^2 + m^2 + t} \right) d^4\bar{l} dt. \quad (10)$$

After the derivative  $\frac{d^2}{dt^2}$  the integral in  $d^4\bar{l}$  is convergent<sup>2</sup>, and this yields

$$\oint \frac{it}{2\pi} (-1 + \log[-t]) \left( -\frac{\pi^2 m^2}{t + m^2} \right) dt \quad (11)$$

The integral over the contour in the complex plane is equal to the residue at the pole  $t \sim -m^2$

$$-\pi^2 (-1 + \log[m^2]) m^4. \quad (12)$$

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<sup>2</sup>We consider only value of  $t$  with non zero imaginary part. See the discussion at the end of this section.

Our final result is correct. The advantage of the procedure above is clear: eqs.(10-11) are well defined and concrete expressions, any step can be done in a pure numerical way. Note that the integral in  $d^4\bar{l}$  must be done after the derivative  $d^2/dt^2$ , and before the integral in  $dt$ ; otherwise the integral in  $d^4\bar{l}$  would be non convergent. In practice, if the above procedure is done numerically, one should replace the integral in  $dt$  with a sum over a finite set of points  $t_i$ ; for the convergence of all integrals, it is enough to choose the  $t_i$  in this set with a non zero imaginary part.

## Method of Integration B

The formula (8) is a transparent and compact expression, which makes manifest some properties of the dimensional regularization; for instance, the invariance under the translations  $\bar{l}_\mu \rightarrow \bar{l}_\mu + p_\mu$  is obvious, since it comes directly from the translational invariance of the integral in  $d^4\bar{l}$ , which is convergent (after the derivative in  $d^2/dt^2$ ). However in certain numerical calculations the (8) could be not efficient. We discuss a different method which is more powerful in practical calculations, even if it requires a slightly more involved formula.

There are several different ways of rewriting the (8); each of them depending on the way we choose to parameterize the virtual momentum  $l_\mu$ . Here we do not intend to make an exhaustive study, we will simply describe a quite general procedure to obtain well defined integral representations for the  $I_k$  in (1).

First we observe that each  $I_k[f]$  is a linear operator

$$I_k[f_1 + f_2] = I_k[f_1] + I_k[f_2] \quad (13)$$

$$I_k[\lambda f] = \lambda I_k[f]. \quad (14)$$

It is quite natural to think that the linearity (13-14) implies that the  $I_k$  are authentic integrals or sum of integrals. In fact we can introduce a simple trick to build concrete integral representations for generic linear functionals.

Consider the space of functions which admits a Laurent expansion around a point  $t_0$ . They are defined by a set of  $a_n$  through the expansion<sup>3</sup>

$$f[t] = \sum_{n=-\infty}^{\infty} a_n (t - t_0)^n. \quad (15)$$

A linear functional  $I[f]$  acting on this set of functions can be written

$$I[f] = I\left[\sum_{n=-\infty}^{\infty} a_n (t - t_0)^n\right] = \sum_{n=-\infty}^{\infty} a_n I[(t - t_0)^n] = \sum_{n=-\infty}^{\infty} a_n b_n \quad (16)$$

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<sup>3</sup>We also assume that the series converges strongly enough to justify the steps below.

where  $b_n = I[(t - t_0)^n]$  is a real (or complex) number. Then we can use the identity

$$\oint_C \frac{1}{2\pi i} \frac{(t - t_0)^n}{(t - t_0)^{k+1}} dt = \delta_{nk} \quad (17)$$

for any  $n, k = 0, \pm 1, \pm 2, \dots$  if the integral contour  $C$  is a small circle, in the complex plane of  $t$ , around the point  $t_0$ . Using the (17), the (16) can be written

$$\begin{aligned} I[f] &= \frac{1}{2\pi i} \oint \left( \sum_{n=-\infty}^{\infty} a_n (t - t_0)^n \right) \left( \sum_{k=-\infty}^{\infty} \frac{b_k}{(t - t_0)^{k+1}} \right) dt = \\ &= \frac{1}{2\pi i} \oint f[t] w[t] dt \end{aligned} \quad (18)$$

with

$$w[t] = \sum_{k=-\infty}^{\infty} \left( \frac{b_k}{(t - t_0)^{k+1}} \right). \quad (19)$$

This is an integral representation of the linear functional  $I$ . This simple trick can be used to build explicit and compact integral representations for any linear functional, whose action on each term of a Laurent expansion is known.

Let us apply it to our problem. We start defining the angular integration in  $n = 4 + \varepsilon$  dimensions. The analytical integration of any tensor with  $k$  (even) indices and constructed with the  $n$  dimensional vector  $l_\mu$  yields<sup>4</sup>

$$\int l_{\bar{\mu}} l_{\bar{\nu}} l_{\bar{\alpha}} \dots l_{\bar{\beta}} l_{\bar{\rho}} l_{\bar{\sigma}} d\Omega_{4+\varepsilon} = 2\pi^2 \frac{(2 + \varepsilon)!!}{(2 + k + \varepsilon)!!} (g_{\bar{\mu}\bar{\nu}} g_{\bar{\rho}\bar{\sigma}} \dots g_{\bar{\alpha}\bar{\beta}} + \dots + g_{\bar{\mu}\bar{\alpha}} g_{\bar{\rho}\bar{\nu}} \dots g_{\bar{\beta}\bar{\sigma}}) l^{2(k/2)} \quad (20)$$

$d\Omega_{4+\varepsilon} = \sin^{2+\varepsilon}[\gamma] \dots \sin^2[\theta] \sin[\phi] d\gamma d\theta d\phi d\delta$  ( $0 < \gamma, \theta, \dots, \phi < \pi; 0 < \delta < 2\pi$ ) is the solid angle. It is understood that the indices ( $\bar{\mu}, \bar{\nu} \dots$ ) above are contracted with some external momenta and/or other tensorial indices (like the spin of the external particles), since the scattering matrix is lorentz invariant. Thus<sup>5</sup> only indices with  $\bar{\mu} \leq 4$  are relevant in our problem. This has been emphasized by the small bar on the Greek indices.

We look for an integral definition which reproduces the (20). To begin the procedure we parameterize  $l_\mu$  with an array of five components<sup>6</sup>

$$l_\mu \rightarrow t \left( x \cos[\theta], x \sin[\theta] \cos[\phi], x \sin[\theta] \sin[\phi] \cos[\delta], x \sin[\theta] \sin[\phi] \sin[\delta], \sqrt{1 - x^2} \right) \quad (21)$$

$\theta, \phi$  and  $\delta$  are the three angles in the space of four dimensions.  $t$  and  $x$  are two complex variables which will be integrated over a complex contour as explained below. The need of the auxiliary

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<sup>4</sup>Here we assume that the overall factor  $\pi^{\frac{\varepsilon}{2}}/\Gamma[\frac{\varepsilon}{2}]$  has been re-absorbed into a redefinition of the scale  $\mu$ , as prescribed by the  $\overline{MS}$  scheme.

<sup>5</sup>We remind that only  $l_\mu$  has components  $> 4$ , and factors  $l_\mu l^\mu = l^2$  are not relevant for the angular integration.

<sup>6</sup>In practice this means that any vector (including external momenta etc.), must be written as an array of five components, but we keep in mind that only  $l_\mu$  has a non zero fifth component.

variable  $x$  will be clear in a moment. Suppose that we integrate over the four dimensional solid angle  $d\Omega_4 = \sin^2[\theta]\sin[\phi] d\theta d\phi d\delta$  all the tensors built with the array (21) in place of an  $n$  dimensional  $l_\mu$ . We get

$$\int l_{\bar{\mu}} l_{\bar{\nu}} l_{\bar{\alpha}} \dots l_{\bar{\beta}} l_{\bar{\rho}} l_{\bar{\sigma}} d\Omega_4 = 2\pi^2 \frac{(2)!!}{(2+k)!!} (g_{\bar{\mu}\bar{\nu}} g_{\bar{\rho}\bar{\sigma}} \dots g_{\bar{\alpha}\bar{\beta}} + \dots + g_{\bar{\mu}\bar{\alpha}} g_{\bar{\rho}\bar{\nu}} \dots g_{\bar{\beta}\bar{\sigma}}) t^k x^k. \quad (22)$$

Comparing this result with the (20) we see that tensorial structure is identical, but the overall factor is not correct. The term of order  $\varepsilon^0$  can be obtained simply setting  $x = 1$  and  $t^2 = l^2$ . Instead for the term of order  $\varepsilon$ , it would be enough to replace  $x^k$  with a suitable factor<sup>7</sup>

$$x^k \rightarrow b_k = \frac{\varepsilon}{2} (1 - \gamma_E - \psi[2 + k/2]) \quad (23)$$

for any even  $k$ .

This can be achieved by means of a linear functional  $I$  such that  $I[x^k] = b_k$ . In fact, following the recipe above (eqs.(15)-(19)) we can build the integral below

$$I[f] = \frac{\varepsilon}{2\pi i} \oint \sum_{k=\text{even}} \frac{b_k}{x^{k+1}} f[x] dx = \frac{\varepsilon}{2\pi i} \oint \frac{x + x^3 \log[1 - \frac{1}{x^2}]}{2(-1 + x^2)} f[x] dx \quad (24)$$

where the integral contour is a circle around the singularity  $x \sim 0$ . The integrand above has a cut in the segment of the real axis between -1 and 1. It is easy to see that in the limit of a path of integration very close to this cut, the logarithm can be approximated by its imaginary part. The real part does not contribute to the full integral, and the logarithm can be replaced by  $\pm i\pi$ . Taking also into account that  $f[x] = f[-x]$  we can rewrite the (24)

$$I[f] = \varepsilon \int_0^1 \left( \frac{x^3}{1 - x^2} \right)_+ f[x] dx. \quad (25)$$

Here the notation  $(\dots)_+$  means that

$$\int_0^1 (h[x])_+ f[x] dx = \int_0^1 h[x] (f[x] - f[1]) dx, \quad (26)$$

and this makes the integral convergent near  $x \sim 1$ .

Finally we are able to write a compact formula for the angular integration in  $4 + \varepsilon$  dimensions

$$\int f[l_\mu] d\Omega_{4+\varepsilon} = \left( \int f[l_\mu] d\Omega_4 \right)_{x=1} + \varepsilon \left( \int_0^1 \int f[l_\mu] \left( \frac{x^3}{1 - x^2} \right)_+ dx d\Omega_4 \right) + \varepsilon^2(\dots). \quad (27)$$

The last step is to define the integration in  $dl^2 = dt^2$ . Again we follow the trick (15)-(19): we have to compute the  $b_\sigma$  in order to get the function  $w[t^2]$ . By definition, we know that for any  $\sigma > 2$

$$b_\sigma = I\left[\frac{1}{(t^2 + m^2)^\sigma}\right] = \int t^{2(1+\varepsilon/2)} dt^2 \frac{1}{(t^2 + m^2)^\sigma} =$$

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<sup>7</sup> The function  $\psi$  is the well known derivative  $d/dx \log[\Gamma(x)]$ ; it comes out when we expand the factorial in the (20) and take only the first order in  $\varepsilon$ .

$$= \frac{1}{(-2 + \sigma)(-1 + \sigma)} \frac{1}{(m^2)^{\sigma-2}} + \varepsilon(\dots), \quad (28)$$

for  $\sigma=1,2$

$$\begin{aligned} b_1 &= 2\frac{m^2}{\varepsilon} + m^2 \log[m^2] + \varepsilon(\dots) \\ b_2 &= -\frac{2}{\varepsilon} - (\log[m^2] + 1) + \varepsilon(\dots) \end{aligned} \quad (29)$$

and for  $\sigma=0,-1,-2,\dots$  the integral is zero ( $b_\sigma = 0$ ). If the integration contour in the  $t^2$  complex plane is a small circle around the pole singularity  $t^2 = -m^2$  and replacing the  $b_n$  in the (19) with the  $b_\sigma$  above we get

$$w[t^2] = \sum_{\sigma=0}^{\infty} b_{\sigma+1} (t^2 + m^2)^\sigma = -t^2 \left( \frac{2}{\varepsilon} + \log[-t^2] \right) \quad (30)$$

and

$$\int t^{2(1+\varepsilon/2)} dt^2 f[t^2] = -\frac{2}{\varepsilon} \left( \frac{1}{2\pi i} \oint f[t^2] t^2 dt^2 \right) - \left( \frac{1}{2\pi i} \oint f[t^2] t^2 \log[-t^2] dt^2 \right) + \varepsilon(\dots). \quad (31)$$

The integral contour is closed, it must contain all the singularities in the complex plane of the function  $f[t^2]$ , except for the singularity in  $t^2 = 0$  and the cut for real and positive  $t^2$  in all integrals where the logarithmic function appears (see for example figure 1). We will comment on this contour of integration later on.

Combing the angular integration (27) and the (31) we get the full formula

$$\int f[l_\mu] d^n l = \int f[l_\mu] t^{2(1+\varepsilon/2)} \frac{dt^2}{2} d\Omega_{4+\varepsilon} = \frac{I_{-1}}{\varepsilon} + I_0 + I_1 \varepsilon + \dots \quad (32)$$

where

$$I_{-1}[f[l_\mu]] = \left( \frac{1}{2\pi i} \oint f[l_\mu] t^2 dt^2 d\Omega_4 \right)_{x=1} \quad (33)$$

$$I_0[f[l_\mu]] = \left( \frac{1}{4\pi i} \oint f[l_\mu] t^2 \log[-t^2] dt^2 d\Omega_4 \right)_{x=1} + \frac{1}{2\pi i} \int_0^1 \oint f[l_\mu] t^2 dt^2 \left( \frac{x^3}{1-x^2} \right)_+ dx d\Omega_4. \quad (34)$$

Clearly the integrals above are well defined and can be evaluated numerically.<sup>8</sup> For example to evaluate the first integral of  $I_0$  in the expression (34), the contour of integration in  $dt^2$  can be chosen as in figure 1. Namely we can divide the contour in three paths. The path A is very close

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<sup>8</sup> Here all integrals are understood to be in Euclidean space. In some kinematical regions, the integration (even if convergent) needs to be regularized by the prescription  $m^2 \rightarrow m^2 + i\varepsilon$ . Also for some infrared singularities the (34) must be rearranged differently or one should use the method A. The practical use of these methods in various realistic situations certainly demands a much more extensive discussion, in this letter we simply present the general idea, a more complete and systematic study will be done elsewhere (see ref.[33]).



to the cut of the logarithm. This function can be replaced with  $\pm i\pi$  (the sign depends if we are above or below the cut). This yields the following integral

$$P_1 = \left( \frac{1}{2} \int_{\Lambda_{ir}^2}^{\Lambda_{uv}^2} f[l_\mu] t^2 dt^2 d\Omega_4 \right)_{x=1}. \quad (35)$$

Note that this is an ordinary four dimensional integral with an infrared and an ultraviolet cut-off:

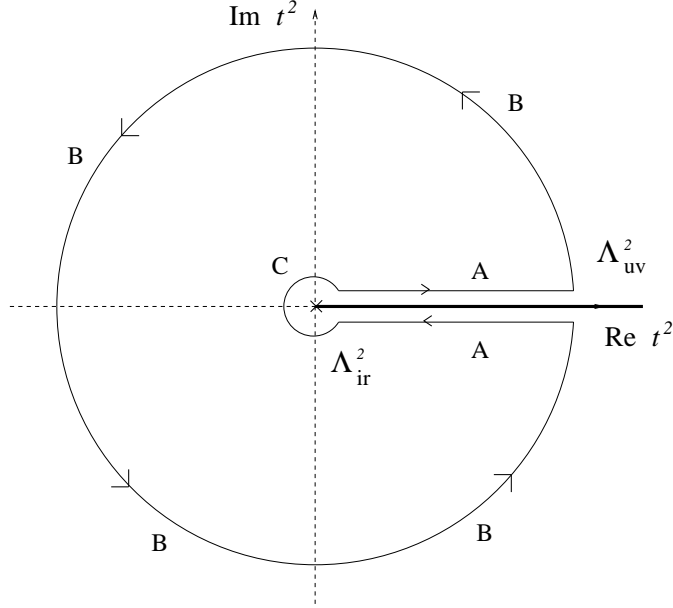


Figure 1: Integration contour in the  $t^2$  complex plane. The thick line in the positive real axis represents the cut of the logarithmic function. The path A is very close to this cut and the logarithm can be replaced there by its imaginary part.  $\Lambda_{uv}^2$  and  $\Lambda_{ir}^2$  are respectively the infrared and ultraviolet cut-off.

in fact after setting  $x = 1$ , the array (21) may be regarded as a genuine four-vector. Then we have the paths B and C, that are two circles with radius  $\Lambda_{uv}^2$  and  $\Lambda_{ir}^2$  respectively

$$P_2 = \left( \frac{1}{4\pi i} \oint_B f[l_\mu] t^2 \log[-t^2] dt^2 d\Omega_4 \right)_{x=1} + \left( \frac{1}{4\pi i} \oint_C f[l_\mu] t^2 \log[-t^2] dt^2 d\Omega_4 \right)_{x=1}. \quad (36)$$

These integrals can be seen as some counterterms which cancel the  $\Lambda_{uv}$  and  $\Lambda_{ir}$  dependence of the integral (35). The sum is equivalent to the finite part ( in the limit  $\varepsilon \rightarrow 0$ ) of the dimensionally regularized integral.

Through the linearity (13-14), this method can be generalized to two (or more) loop calculations. If  $q_\mu$  and  $l_\mu$  are the two loop virtual momenta, then we can rewrite the integration variables  $d^n l = l^{2(1+\varepsilon/2)} dl^2 d\Omega_{4+\varepsilon}/2$  and  $d^n q = dq_\parallel q_\perp^{2(1+\varepsilon/2)} dq_\perp d\Omega_{3+\varepsilon}$ .  $q_\parallel$  is the component of  $q_\mu$  parallel to  $l_\mu$ ,  $q_\perp$  is the total length of the components of  $q_\mu$  orthogonal to  $l_\mu$ . One then defines two six dimensional vectors  $l_\mu$  and  $q_\mu$ , analogously to (21), as functions of five angles  $\theta, \phi, \delta, \phi', \delta'$ , two

auxiliary variables  $x$  and  $y$  (needed for the integrations  $d\Omega_{4+\varepsilon}$  and  $d\Omega_{3+\varepsilon}$ ) and the complex variables  $l^2, q_{\parallel}, q_{\perp}$ . Applying the trick (15)-(19) one obtains the function  $w[l^2, q_{\parallel}, q_{\perp}]$ . Needless to say, this contains dilogarithms in addition to some logarithms. From this guidelines, one gets the analogue of the (34), for two loop calculations [33].

## Some Numerical Examples

In order to make clear the practical purposes of these methods, we discuss here some very simple examples. Suppose that we want to integrate

$$\int d^n l \frac{1}{(l+p)^2 + m^2} \quad (37)$$

with  $m^2 = 2$  and  $p^2 = 1$ . Our numerical approach must reproduce the analytical result

$$2\frac{\pi^2}{\varepsilon}m^2 + \pi^2 m^2 \log[m^2] + \dots \quad (38)$$

Suppose that we want to extract the finite part<sup>9</sup>  $\pi^2 m^2 \log[m^2]$ . We apply the second method B described above. We use the five dimensional array (21) for  $l_{\mu}$ . Then the integrand is a function of  $x, t^2, \theta, \phi, \delta$ . The finite part of the (37) is obtained computing the two integrals in  $I_0$  (eq.(34)). For the first one we choose the contour of integration as in figure 1: the numerical values of  $\Lambda_{uv}$  and  $\Lambda_{ir}$  have to be chosen in such a way that the contour A+B+C contains all the physical singularities in the complex  $t^2$  plane. Since the integral is infrared convergent we can take the limit  $\Lambda_{ir} \rightarrow 0$ . Instead, in the ultraviolet region, we can cut the integral at  $\Lambda_{uv}^2 = 9$ . We also must set  $x = 1$ . Then the integral (35) becomes

$$P_1 = \int_0^9 dt^2 d\Omega_4 \frac{1}{2} \frac{t^2}{t^2 + 2t \cos[\theta] + 3} = 51.929. \quad (39)$$

One can recognize in the integral above the ordinary four dimensional integration with a cut-off  $\Lambda_{uv}^2$ . Then the integration in  $dt^2$  follows the path B

$$P_2 = \frac{1}{2\pi i} \oint_B dt^2 d\Omega_4 \log[-t^2] \frac{1}{2} \frac{t^2}{t^2 + 2t \cos[\theta] + 3} = -43.1815. \quad (40)$$

This completes the first integral in  $I_0$ . The second one in eq.(34) contains a non trivial integration in the variable  $x$ , which must be performed using the prescription (26). The contour of integration in  $dt^2$  is rather simple: there is no logarithm, and no cut in the real axis; then we do not need to follow the path A. There is no infrared singularity and also the path C vanishes. The only non trivial contribution comes from the path B, a circle of radius  $\Lambda_{uv}^2$ . This gives

$$P_3 = \frac{1}{i2\pi} \int_0^1 \oint_B \frac{1}{t^2 + 2xt \cos[\theta] + 3} t^2 dt^2 \left( \frac{x^3}{1-x^2} \right)_+ dx d\Omega_4 = 4.935. \quad (41)$$

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<sup>9</sup> It is more instructive to discuss the finite part, since the integral contour in (33) is trivial, and extracting the singular part is straightforward.

Note in the denominator the appearance of the variable  $x$ , which comes out when we take the scalar product  $l \cdot p$  with  $l_\mu$  from eq.(21) and  $p^\mu$  in the direction  $\mu = 1$ . The sum of the three contributions yields

$$I_0[\frac{1}{(l+p)^2 + m^2}] = P_1 + P_2 + P_3 = 13.6825 \simeq \pi^2 m^2 \log[m^2], \quad (42)$$

in perfect agreement with the analytical expression. The method can also be applied for infrared divergent integrals. The integral

$$\int d^n l \frac{1}{l^6} \frac{1}{(l+p)^2 + m^2} \quad (43)$$

is ultraviolet convergent and infrared divergent. Thus we choose  $\Lambda_{uv} = \infty$  and  $\Lambda_{ir} = 1/4$ . We get

$$P_1 = \int_{1/4}^{\infty} t^2 dt^2 \frac{1}{2} d\Omega \frac{1}{t^2 + 2t \cos[\theta] + 3} \frac{1}{t^6} = 10.843, \quad (44)$$

$$P_2 = \frac{1}{2\pi i} \oint_C dt^2 d\Omega_4 \log[-t^2] \frac{1}{2t^4} \frac{1}{t^2 + 2t \cos[\theta] + 3} = -12.1255$$

$$P_3 = \frac{1}{i2\pi} \int_0^1 \oint_B \frac{1}{t^2 + 2xt \cos[\theta] + 3} t^2 dt^2 \left( \frac{x^3}{1-x^2} \right)_+ dx d\Omega_4 = -0.1825. \quad (45)$$

The sum of the three integrals gives

$$I_0[\frac{1}{l^6} \frac{1}{(l+p)^2 + m^2}] = P_1 + P_2 + P_3 = -1.465, \quad (46)$$

very close to the exact result

$$-\frac{1}{27}\pi^2 \left(1 + \log\left[\frac{81}{4}\right]\right). \quad (47)$$

## Conclusions

In the dimensional regularization of loop integrals, usually one defines the integral as a function of the number of dimensions in a region of  $n$  where the integral is convergent. Then one makes an analytic continuation to obtain a definition of the integral also in regions of  $n$  where the integral is divergent. This procedure is clear and unambiguous, however it can only be applied in pure analytic (and symbolic) calculations. It cannot be converted into an obvious numerical procedure. One is obliged to perform symbolic calculations, which sometimes becomes lengthy and/or unaffordable. In this paper we have shown how to set up a different approach, where each coefficient the Laurent expansion in powers of  $\varepsilon$  in eq. (1) can be written in terms of concrete and convergent integrals (well) defined in a five dimensional space (see eq.(8) and eqs.(32),(33) and

(34) . Instead of abstract  $n$  dimensional vectors we have to deal with “concrete” five dimensional vectors, and we have to perform integrations over a compact space. This formulation has the advantage to allow us the evaluation of all integrals through pure numerical methods. Clearly further work is needed, to generalize the above result to more loops and to prove the efficiency of this technique in realistic physical problems. However we believe that the simplicity of the numerical approach makes it very promising.

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